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A New Method for Solving Fuzzy Bernoulli Differential Equation

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Abstract. In the present paper, a new method is proposed to solve fuzzy Bernoulli differential equation (FBDE) of the form $\tilde{u}'(t)+p(t)\ \tilde{u}(t) = q(t)\,\tilde{u}^n(t)\,, \quad t \geq 0,\ t \in R,\ \tilde{u}(0) = \tilde{u}_0$, where $p(t)$ and $q(t)$ are real continues functions, $\tilde{u}(t) = (x\,(t), y\,(t), z\,(t))$ is LR fuzzy function, $\tilde{u}_0$ is LR fuzzy number and $n \neq 0$, 1. At the beginning, nth power of LR fuzzy number is defined. Thereafter, [i.gH]-differentiability and [ii.gH]-differentiability are described using generalized Hukuhara difference and differentiability. At the end, $\tilde{u}\,(t)$ is determined as a LR fuzzy function through solving 1-cut FBDE, finding the sign of $x\,(t)$, $p(t)$, $q(t)$ and applying the proposed theorem. Also, numerical examples are presented to verify the effectiveness of the proposed method.

AMS Subject Classification: MSC 34A07

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1 Introduction

The dynamic characteristics of many real word problems are not constant for several reasons and varies as time elapses. The best tool for defining these variations are differential equations. In this regard, fuzzy differential equations emerge in which variables involve ambiguity and errors. Over the last several decades, many different investigations have been performed in the field of fuzzy differential equations [1-8]. For example, Hukuhara derivative of fuzzy equation was proposed in 1983 [9]. In 2005,B. Bede and et. al. proposed a method to determine weak and strong generalized derivative of fuzzy functions and introduced its applications [10].
In 2007, the problem of how fuzzification of different formulations of the same crisp equation translate into several inequivalent FDEs was investigated [11]. Also, it was shown that the characteristics and behavior of the solutions of different fuzzy versions of equivalent crisp ODEs are very different.In 2013, under strong generalized differentiability [12] the series solution of fuzzy differential equation was found. In 2015, the approximate solution of fuzzy equations was calculated by Otadi and Mosleh under generalized differentiability. Finally, in 2018,under the concept of generalized differentiability and using new Runge-Kutta – like formula of order 4, numerical solution of fuzzy differential equations was determined [13,14].
In the present paper, a new simple and effective method is proposed in which the solution of fuzzy Bernoulli differential equation is determined using operations on LR fuzzy numbers and generalized Hukuhara difference and derivative.
The rest of the paper is as follows: Basic necessary definitions are presented in section 2. The proposed method for solving fuzzy differential equation is introduced in section 3. Numerical examples are involved in section 4 and finally conclusion is presented in section 5.

2 Basic definitions

Definition 2.1. [15] *A fuzzy number $\tilde{M}$ is called LR-type if there exist reference functions L (for left), R (for right), and scalers $\alpha > 0$, $\beta > 0$*

such that:

$$\mu_{\tilde{M}}(x) = \begin{cases} L\ \left(\frac{m-x}{\alpha}\right) & for\ x \leq m\,, \\ R\ \left(\frac{x-m}{\beta}\right) & for\ x \geq m\,. \end{cases}$$

where m *is a real number that is said to the mean value of* $\tilde{M}$*.* α *and* β *denote the left and the right spread, respectively. Symbolically,* $\tilde{M}$ *is described by*(m, α, β)*.*

Definition 2.2. [15,16] *Consider* $\tilde{M}$, $\tilde{N}$*as two fuzzy numbers of LR-type:*

$$\tilde{M} = (m, \alpha, \beta), \quad \tilde{N} = (n, \gamma, \delta)$$

Then

$$(m, \alpha, \beta) + (n, \gamma, \delta) = (m + n, \alpha + \gamma, \beta + \delta),$$

$$\lambda\,.\,(m, \alpha, \beta) = \begin{cases} (\,\lambda\, m,\ \lambda\, \alpha,\ \lambda\, \beta) & \lambda \geq 0, \\ (\,\lambda\, m,\ -\lambda\, \beta,\ -\lambda\, \alpha) & \lambda < 0, \end{cases}$$

$$(m, \alpha, \beta)\ .\ (n, \gamma, \delta) \approx \begin{cases} (m\, n, m\, \gamma + n\, \alpha, m\, \delta + n\, \beta) & \tilde{M},\, \tilde{N} > 0, \\ (m\, n, n\alpha - m\, \delta, n\, \beta - m\gamma) & \tilde{M} < 0,\, \tilde{N} > 0, \\ (m\, n,\ m\, \gamma - n\, \beta,\ m\, \delta - n\, \alpha) & \tilde{M} > 0,\, \tilde{N} < 0, \\ (m\, n, -n\, \beta - m\, \delta, -n\, \alpha - m\, \gamma) & \tilde{M},\, \tilde{N} < 0. \end{cases}$$

Definition 2.3. [17] *The generalized Hukhara difference of two fuzzy numbers is determined as follow:*

$$u -_{gH} v = w \Leftrightarrow \begin{cases} & (i)\ u = v + w, \\ or & (ii)\ v = u + (-1)\ w. \end{cases}$$

Definition 2.4. [18] *If* $x_0 \in (a, b)$*and* h *is such that* $x_0 + h \in (a, b)$*, then the gH-derivative of a function* $f : (a, b) \to I$ *can be described as follow:*

$$f'_{gH}(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) -_{gH} f(x_0)}{h}.$$

3 Solving fuzzy Bernoulli differential equation

Among various important problems in the field of mathematics, solving differential equations especially fuzzy Bernoulli differential equations can be inferred as one of the most important problems. Therefore, in the following a new method is introduced to solve such problems. The proposed method is shown to be simple and practical in use.

Definition 3.1. *The function $\tilde{u}(t)$ is called LR fuzzy function if its output are LR fuzzy numbers for each t. The function is shown by $\tilde{u}(t) = (x\,(t), y\,(t), z\,(t))$ where $x\,(t)$ is the core function and $y\,(t) and z\,(t)$ are the left and the right spread functions, respectively. Moreover, it can be said that the fuzzy function $\tilde{u}(t)$ is positive (negative) if $x\,(t) > 0$ $(x\,(t) < 0)$.*

Definition 3.2. *Consider the differential equation as*

$$\tilde{u}'(t) + p(t)\,\tilde{u}(t) = q(t)\;\; \tilde{u}^n(t)\,, \qquad t \geq 0,\; t \in R \tag{1}$$

with the initial value as

$$\tilde{u}(0) = \tilde{u}_0$$

The aforementioned differential equation is said to be fuzzy Bernoulli differential equation (FBDE) where $p(t)$ and $q(t)$ are real continues functions, $\tilde{u}(t)$ denotes a LR fuzzy function, $\tilde{u}_0$ refers to a LR fuzzy number and $n \neq 0, 1$.
To solve the problem of FBDE (1), the n^{th} power of LR fuzzy function is required to be defined. Therefore, a theorem is defined in the following to determine $\tilde{u}^n(t)$.

Theorem 3.3. *Assume that $\tilde{u}(t) = (x(t), y(t), z(t))$ is a LR fuzzy function*

a) If $\tilde{u}(t) > 0$, then

$$\tilde{u}^n(t) = \left(x^n(t), n\, x^{n-1}(t)y(t), n\, x^{n-1}(t)z(t)\right). \tag{2}$$

b) If $\tilde{u}(t) < 0$, *then*

$$u^n(t) = \begin{cases} \left(x^n(t), n\ x^{n-1}(t)\ y(t), n\ x^{n-1}(t)\ z(t)\right) & n \text{ odd}, \\ \\ \left(x^n(t), -n\ x^{n-1}(t)\ z(t), -n\ x^{n-1}(t)\ y(t)\right) & n \text{ even}. \end{cases} \tag{3}$$

Proof.

a) With the help of definition 2.2

$$\begin{aligned} \tilde{u}^2(t) &= (x\,(t), y\,(t), z\,(t))\ .\ (x\,(t), y\,(t), z\,(t)) \\ &= \left(\,x^2(t),\ 2\,x^2(t)\,y(t),\ 2x^2(t)\,z(t)\,\right), \end{aligned}$$

$$\begin{aligned} \tilde{u}^3(t) &= \tilde{u}^2(t)\,.\,u\,(t) \\ &= \left(x^2(t),\ 2\,x^2(t)\,y(t),\ 2x^2(t)\,z(t)\right)\ .\ (x\,(t),\ y\,(t),\ z\,(t)) \\ &= \left(\,x^3(t), 3\,x^2(t)\,y\,(t), 3\,x^2(t)\,z\,(t)\,\right), \end{aligned}$$

Now, by continuing this procedure to the n^{th} step, the equation in the following can be achieved:

$$\tilde{u}^n(t) = \left(x^n(t), n\,x^{n-1}(t)y(t), n\,x^{n-1}(t)z(t)\right)\ .$$

b) The procedure of the proof in part(b) is similar to that in part(a) and hence is omitted.

□

Definition 3.4. *Consider the LR fuzzy function* $\tilde{u}(t)$

a) $\tilde{u}(t)$ *is [i.gH]-differentiable, If* $\tilde{u}(t)$ *is gH-differentiable and* $\tilde{u}(t+h) -_{gH} \tilde{u}(t)$ *is calculated from part (i) of definition 2.3.*

b) $\tilde{u}(t)$ *is [ii.gH]-differentiable,If* $\tilde{u}(t)$ *is gH-differentiable and* $\tilde{u}(t+h) -_{gH} \tilde{u}(t)$ *is calculated from part (ii) of definition 2.3.*

Now, a theorem is introduced that enables us to determine the first order derivative of LR fuzzy function using [i.gH]-differentiability and [ii.gH]-differentiability.

Theorem 3.5. *If $\tilde{u}(t) = (x(t), y(t), z(t))$ is a LR fuzzy function:*

a) If $\tilde{u}(t)$ is [i.gH]-differentiable then $\tilde{u}'(t) = (x'(t), y'(t), z'(t))$.

b) If $\tilde{u}(t)$ is [ii.gH]-differentiable then $\tilde{u}'(t) = (x'(t), -z'(t), -y'(t))$.

Proof. The proof of part (b) is described in the following:

$$
\begin{aligned}
\tilde{u}'(t) &= \lim_{h\to 0} \frac{u\,(t+h) -_{gH} u\,(t)}{h} \\
&= \lim_{h\to 0} \frac{(x\,(t+h) - x\,(t), z\,(t) - z\,(t+h), y\,(t) - y\,(t+h))}{h} \\
&= \lim_{h\to 0} \left(\frac{x\,(t+h) - x\,(t)}{h}, -\frac{z\,(t+h) - z\,(t)}{h}, -\frac{y\,(t+h) - y\,(t)}{h} \right) \\
&= (x'(t), -z'(t), -y'(t)).
\end{aligned}
$$

Now for solving FBDE problem a theorem is proposed in the following that solves the problem using 1-cut system and determining the sign. □

Theorem 3.6. *The solution of fuzzy Bernoulli differential equation with the initial value (1), $\tilde{u}(t) = (x(t), y(t), z(t))$ is as follows*

A) A1) [i.gH]-differentiability, $x\,(t) > 0$ or $x\,(t) < 0\,,$ n odd, $p\,(t) > 0$, $q\,(t) > 0$.

A2) [i.gH]-differentiability,

$$x\,(t) < 0\,,\ n\ even, p\,(t) > 0,\ q\,(t) < 0.$$

A3) [ii.gH]-differentiability,

$$x\,(t) < 0\,,\ n\ even, p\,(t) < 0,\ q\,(t) > 0.$$

A4) [ii.g]-differentiability, $x\,(t) > 0$ or $x\,(t) < 0\,,$ n odd, $p\,(t) < 0$, $q\,(t) < 0$.

$$
\begin{aligned}
\tilde{u}\,(t) = \Big(&\Big(e^{(n-1)\int p\,(t)\,dt} \Big(\int (1-n)\,q\,(t)\,e^{(1-n)\int p\,(t)\,dt}\,dt + c_1 \Big) \Big)^{\frac{1}{1-n}}, \\
&c_2\,e^{-\int p\,(t)\,dt}\, e^{n\int q\,(t)\,x^{n-1}(t)\,dt},\ c_3\,e^{-\int p\,(t)\,dt}\, e^{n\int q\,(t)\,x^{n-1}(t)\,dt} \Big),
\end{aligned}
\tag{4}
$$

B) B1) [i.gH]-differentiability,

$$x(t) < 0\,,\; n\ even, p(t) > 0,\; q(t) > 0.$$

B2) [i.gH]-differentiability, $x(t) > 0$ *or* $x(t) < 0\,,\; n$ *odd,* $p(t) > 0,\; q(t) < 0$.

B3) [ii.gH]-differentiability, $x(t) > 0$ *or* $x(t) < 0\,,\; n$ *odd,* $p(t) < 0,\; q(t) > 0$.

B4) [ii.gH]-differentiability,

$$x(t) < 0\,,\; n\ even, p(t) < 0,\; q(t) < 0.$$

$$\begin{aligned}
\tilde{u}(t) = \Bigg(& \left(e^{(n-1)\int p(t)\,dt} \left(\int (1-n)\, q(t)\, e^{(1-n)\int p(t)\,dt}\, dt + c_1 \right) \right)^{\frac{1}{1-n}}, \\
& e^{-\int p(t)\,dt}\, e^{n\int q(t)\, x^{n-1}(t)\,dt} \left(\int -n\; q(t)\, x^{n-1}(t) \right. \\
& \left(c_2\, e^{-\int p(t)\,dt}\, e^{-n\int q(t)\; x^{n-1}(t)\,dt} \right) \\
& \left. \int\; e^{\int p(t) - n\; q(t)\; x^{n-1}(t)\,dt} dt + c_3 \right), \\
& e^{-\int p(t)\,dt}\, e^{n\int q(t)\; x^{n-1}(t)\,dt} \left(\int -n\; q\;(t)\; x^{n-1}(t) \right. \\
& \left. \left(c_2\, e^{-\int p(t)\,dt}\, e^{-n\int q(t)\, x^{n-1}(t)\,dt} \right)\; e^{\int p(t) - n\; q(t)\; x^{n-1}(t)\;dt} dt + c_4 \right) \Bigg),
\end{aligned} \tag{5}$$

C) C1) [i.gH]-differentiability, $x(t) > 0$ *or* $x(t) < 0\,,\; n$ *odd,* $p(t) < 0,\; q(t) > 0$.

C2) [i.gH]-differentiability,

$$x(t) < 0\,,\; n\ even, p(t) < 0,\; q(t) < 0.$$

C3) [ii.gH]-differentiability,

$$x(t) < 0\,,\; n\ even, p(t) > 0,\; q(t) > 0.$$

C4) [ii.gH]-differentiability, $x\,(t) > 0 \quad or \quad x\,(t) < 0\,,\ n$ *odd,* $p\,(t) > 0,\ q\,(t) < 0.$

$$\tilde{u}\,(t) = \Bigg(\Bigg(e^{(n-1)\,\int p\,(t)\,dt}\Bigg(\int (1-n)\,q\,(t)\, e^{(\,1-n\,)\,\int\,p\,(t)\,dt}\,dt + c_1\Bigg)\Bigg)^{\frac{1}{1-n}},\ e^{-\int p\,(t)\,dt}\,e^{n\,\int q\,(t)\,x^{n-1}(t)\,dt}\Bigg(\int p\,(t)\ \Big(c_2\,e^{\int p\,(t)\,dt}\,e^{n\int q\,(t)\,x^{n-1}(t)\,dt}\Big)\, e^{\int p\,(t)-n\,q\,(t)\,x^{n-1}(t)\,dt}\,dt + c_3\Bigg), e^{-\int p\,(t)\,dt}\,e^{n\,\int q\,(t)\,x^{n-1}(t)\,dt}\Bigg(\int p\,(t)\ \Big(c_2\,e^{\int p\,(t)\,dt}\,e^{n\int q\,(t)\,x^{n-1}(t)\,dt}\Big)\ e^{\int p\,(t)-n\,q\,(t)\,x^{n-1}(t)\,dt}dt + c_4\Bigg)\Bigg)\ , \tag{6}$$

D) D1) [i.gH]-differentiability,

$$x\,(t) < 0\,,\ n\ even, p\,(t) < 0,\ q\,(t) > 0.$$

D2) [i.gH]-differentiability, $x\,(t) > 0 \quad or \quad x\,(t) < 0\,,\ n$ *odd,* $p\,(t) < 0,\ q\,(t) < 0.$

D3) [ii.gH]-differentiability,$x\,(t) > 0 \quad or \quad x\,(t) < 0\,,\ n$ *odd,* $p\,(t) > 0,\ q\,(t) > 0.$

D4) [ii.gH]-differentiability,

$$x\,(t) < 0\,,\ n\ even, p\,(t) > 0,\ q\,(t) < 0.$$

$$\tilde{u}\,(t) = \Bigg((e^{(n-1)\,\int p\,(t)\,dt}(\int (1-n)\,q\,(t)\,e^{(\,1-n\,)\,\int\,p\,(t)\,dt}\,dt + c_1)\,)^{\frac{1}{1-n}}, e^{-\int p\,(t)\,dt}\,e^{n\,\int q\,(t)\,x^{n-1}(t)\,dt}\ (\int (\,p\,(t) - n\ q\,(t)\,x^{n-1}(t)\ )\,(\,c_2\,e^{\int p\,(t)\,dt}\,e^{-n\,\int q\,(t)\,x^{n-1}(t)\,dt}\,)\, e^{\int p\,(t)-n\,q\,(t)\,x^{n-1}(t)\,dt} + c_3\,)\,, e^{-\int p\,(t)\,dt}\,e^{n\,\int q\,(t)\,x^{n-1}(t)\,dt}\,(\int (\,p\,(t) - n\,q\,(t)\,x^{n-1}(t)\ )\,(\,c_2\,e^{-\int p\,(t)\,dt}\,e^{-n\int q\,(t)\,x^{n-1}(t)\,dt}\,)\ e^{\int p\,(t)-n\,q\,(t)\,x^{n-1}(t)\,dt}dt + c_4\,)\Bigg)\ . \tag{7}$$

where the constants in (4-7) can be specified using initial values.

Proof.

A) In A1 to A4, by substituting $\tilde{u}(t) = (x\,(t), y\,(t), z\,(t))$ in (1) and applying definition2.3, theorems 3.1 and 3.2, the following equations set is gained:

$$\begin{cases} x'\,(t) + p\,(t)\,x\,(t) = q\,(t)\;x^{n}(t)\,, \\ y'(t) + p\,(t)\,y\,(t) = n\,q\,(t)\,x^{n-1}(t)\,y\,(t)\,, \\ z'(t) + p\,(t)\,z\,(t) = n\,q\,(t)\,x^{n-1}(t)\,z\,(t)\,, \end{cases} \tag{8}$$

where the first equation is the Bernoulli differential equation and the second and the third equations are linear first order one. Therefore:

$$\begin{aligned} x\,(t) &= \left(e^{(n-1)\int p\,(t)\,dt}\;(\int (1-n)\,q\,(t)\,e^{(1-n)\int p\,(t)\,dt}\,dt + c_1\,)\right)^{\frac{1}{1-n}}, \\ y\,(t) &= c_2\,e^{-\int p\,(t)\,dt}\;e^{n\int q\,(t)\,x^{n-1}(t)\,dt}, \\ z\,(t) &= c_3\,e^{-\int p\,(t)\,dt}\;e^{n\int q\,(t)\,x^{n-1}(t)\,dt}, \end{aligned} \tag{9}$$

B) In B1 to B4, by substituting $\tilde{u}(t) = (x\,(t), y\,(t), z\,(t))$ in (1) and with the help ofdefinition2.3, theorems3.1 and 3.2, the following differential equations set can be achieved:

$$\begin{cases} x'(t) + p\,(t)\,x\,(t) = q\,(t)\,x^{n}(t)\,, \\ y'(t) + p\,(t)\,y(t) = -n\,q\,(t)\,x^{n-1}(t)\,z\,(t)\,, \\ z'(t) + p\,(t)\,z\,(t) = -n\,q\,(t)\,x^{n-1}(t)\,y\,(t)\,, \end{cases} \tag{10}$$

likewise, $x\,(t)$ is achieved from (9) and for calculating$y\,(t)$ and $z\,(t)$ the following equation is defined:

$$w\;(t) = y\,(t) + z\,(t)\,, \tag{11}$$

Now, the following equation is obtained through adding the second and the third equation of (10) and substituting $w\,(t)$:

$$w'\,(t) + (p\,(t) + n\,q\,(t)\,x^{n-1}(t)\,)\,w\,(t) = 0\,,$$

As a result:

$$w\,(t) = c_2\, e^{-\int p\,(t)\, dt}\, e^{-n\int q\,(t)\, x^{n-1}(t)\, dt}\,,$$

Now, by substituting (11) in the second equation of (10), the following equation can be achieved:

$$y'\,(t) + (p\,(t) - n\, q\,(t)\, x^{n-1}(t)\,)\, y\,(t) = -n\, q\,(t)\, x^{n-1}(t)\ w\,(t)\,,$$

Therefore:

$$y(t) = e^{-\int p\,(t)\, dt}\, e^{n\int q\,(t)\, x^{n-1}(t)\, dt}\,(\ \int (\ -n\, q\,(t)\, x^{n-1}(t)\, w\,(t)\ )$$
$$e^{\int p\,(t)\, dt}\, e^{-\int n\, q\,(t)\, x^{n-1}(t)\, dt}\, dt + c_2\ )\ ,$$

Similarly, it can be achieved that:

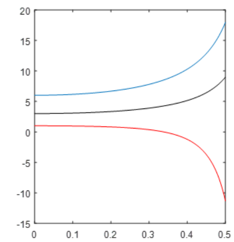
$$z\,(t) = e^{-\int p\,(t)\, dt}\, e^{n\int q\,(t)\, x^{n-1}(t)\, dt}\,(\ \int (\ -n\, q\,(t)\, x^{n-1}(t)\, w\,(t)\ )$$
$$e^{\int p\,(t)\, dt}\, e^{-\int n\, q\,(t)\, x^{n-1}(t)\, dt}\, dt + c_3\ )\ .$$

For the rest of the cases, the proof can be done similarly.
According to theorem 3.3, if the initial value of fuzzy Bernoulli differential equation (1) is symmetric, and FBDE has a solution then it can be easily found that the solution will be a fuzzy symmetric as $u\,(t) = (x(t), \alpha(t), \alpha(t))$. This case is investigated in example 4.3. □
Summary of the method: The procedure of the proposed method for solving equation (1) is briefly described in the following:
In the first step, 1-cut of the system is solved and the sign of $x(t)$,$p\,(t)$ and$q\,(t)$ are determined. In the second step, the solution of [i.gH]-differentiable and [ii.gH]-differentiable is obtained using theorem 3.3. In the third step, the solution which is closer to the decision maker is considered as the solution of equation (1). It must be mentioned that the fuzzy Bernouli differential equation will not have fuzzy solution if for each $t \geq 0$, $y\,(t)$, $z\,(t)$ are negative or $y\,(t) \geq 0$ in $[a, b]$, $z\,(t) \geq 0$ in $[c, d]$, where $[a, b] \bigcap [c, d] = \phi$.

4 Numerical examples

In order to show the performance and applicability of the proposed method, some numerical examples are presented in this section. It is also found that the proposed method is simple in use and application.

Figure 1: [i.gH]-differentiable solution of equation (12).



Example 4.1. Let a fuzzy Bernoulli differential equation be as follow:

$$u'(t) + t\,\tilde{u}\,(t) = 2\,t\,\tilde{u}^2(t) \qquad \tilde{u}(0) = (3, 2, 1)\,, \tag{12}$$

In the first step, the following equation can be obtained by solving 1-cut of the system:

$$x\,(t) = 3\;(\;6 - 5e^{\frac{t^2}{2}}\;)^{-1}\;,$$

It can be seen that $x(t)$ is continues and positive in $[0, 0.5]$ and $p\,(t) > 0, q\,(t) > 0$, as a result in the second step [i.gH]-differentiability of theorem 3.3 in part A1 is used to have:

$$y\,(t) = 2\,e^{\frac{t^2}{2}}\,(\;6\;-5\,e^{\frac{t^2}{2}}\;)^{-2}\,,$$

$$z\,(t) = e^{\frac{t^2}{2}}\,(\;6\;-5\,e^{\frac{t^2}{2}}\;)^{-2}\,,$$

It is found that $y\,(t) > 0, z\,(t) > 0$ in $[0, 0.5]$, therefore; the solution of fuzzy Bernoulli differential equation (12) with [i.gH]-differentiability can be achieved as

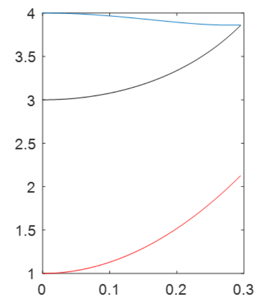
$$\tilde{u}\,(t) = \left(3\;(\;6\;-5\,e^{\frac{t^2}{2}}\;)^{-1}\,, 2\;e^{\frac{t^2}{2}}\,(\;6\;-5\,e^{\frac{t^2}{2}}\;)^{-2}\,, e^{\frac{t^2}{2}}\,(\;6\;-5\,e^{\frac{t^2}{2}}\;)^{-2}\right)\;,$$

here, the solution is illustrated in Figure 1

Similar to the previous case; using [ii.gH]-differentiability of theorem 3.3 in part D3, the following spreads can be obtained in $[0, 0.295]$:

$$y\,(t) = e^{\frac{t^2}{2}}\,(\;6 - 5\,e^{\frac{t^2}{2}}\;)^{-2}\,(\;1944e^{-t^2} - 6480e^{-\frac{t^2}{2}} - 4500e^{\frac{t^2}{2}} + \frac{1875}{2}e^{t^2} + 8100.5\;)\;,$$

Figure 2: [ii.gH]-differentiable solution of equation (12).



$z\,(t) = e^{\frac{t^2}{2}}\,(\,6\,-5\,e^{\frac{t^2}{2}}\,)^{-2}\,(\,1944e^{-t^2} - 6480e^{-\frac{t^2}{2}} - 4500e^{\frac{t^2}{2}} + \frac{1875}{2}e^{t^2} + 8099.5\,)$,

Hence, the solution of fuzzy Bernoulli differential equation (12) using [ii.gH]-differentiability is determined as follows:

$$\tilde{u}\,(t) = \Big(3\,(\,6\,-5\,e^{\frac{t^2}{2}}\,)^{-1}, e^{\frac{t^2}{2}}\,(\,6\,-5\,e^{\frac{t^2}{2}}\,)^{-2}\,(\,1944\,e^{-t^2} - 6480\,e^{-\frac{t^2}{2}} - 4500\,e^{\frac{t^2}{2}} + \frac{1875}{2}e^{t^2} +8100.5\,),\; e^{\frac{t^2}{2}}\,(\,6\,-5\,e^{\frac{t^2}{2}}\,)^{-2}\,(\,1944\,e^{-t^2} - 6480\,e^{-\frac{t^2}{2}} - 4500e^{\frac{t^2}{2}} + \frac{1875}{2}e^{t^2} + 8099.5\,)\Big)\,.$$

The solution is depicted in Figure 2.

Example 4.2. Assumea fuzzy Bernoulli differential equation as follow:

$$u'(t) - 3\,t^{\,2}\tilde{u}\,(t) = -t^2\,\tilde{u}^3(t) \qquad \tilde{u}(0) = (6, 2, 1)\,, \tag{13}$$

The solution of solving 1-cut of the system is as follow:

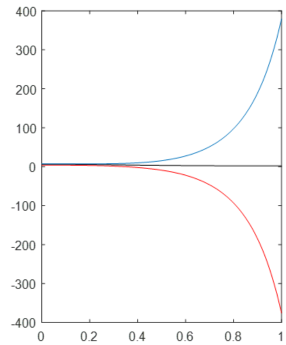
$$x\,(t) =\; 6\,e^{t^3}(12e^{2t^3} - 11)^{-\frac{1}{2}}\,,$$

It can be seen that $x(t)$is positive and continues and $p\,(t) < 0, q\,(t) < 0$. Therefore, in the second step, applying [i.gH]-differentiability of theorem 3.3 part D2, the following spreads can be achieved:

$$y\,(t) = e^{t^3}\,(\,12\,e^{2t^3} - 11\,)^{-\frac{3}{2}}\,(\,-\frac{3993}{2}e^{-2t^3} - 7128e^{2t^3} + 2592\,e^{4t^3} + 6534.5\,)\,,$$

$$z\,(t) = e^{t^3}(12\,e^{2t^3} - 11)^{-\frac{3}{2}}(-\frac{3993}{2}e^{-2t^3} - 7128e^{2t^3} + 2592\,e^{4t^3} + 6533.5),$$

Figure 3: [i.gH]-differentiable solution of equation (13).



It is found that $y(t) > 0, z(t) > 0$. As a result, the solution of fuzzy Bernoulli differential equation (13) with [i.gH]-differentiability can be calculated as follow:

$$\tilde{u}(t) = \Big(6\,e^{t^3}\,(\,12e^{2t^3}-11\,)^{-\frac{1}{2}}, e^{t^3}\,(\,12\,e^{2t^3}-11\,)^{-\frac{3}{2}}\,(\,-\frac{3993}{2}e^{-2\,t^3}-7128\,e^{2\,t^3} + 2592\,e^{4\,t^3}+6534.5\,)\,,\, e^{t^3}\,(\,12\,e^{2t^3}-11\,)^{-\frac{3}{2}}\,(\,-\frac{3993}{2}e^{-2\,t^3}-7128\,e^{2\,t^3} + 2592\,e^{4\,t^3}+6533.5\,)\Big)\,,$$

which is shown in Figure 3. In the same way, the following spreads can be obtained through applying [ii.gH]-differentiability of theorem 3.3 part A4:

$$y(t) = 2\,e^{t^3}\,(\,12\,e^{2t^3}-11\,)^{-\frac{3}{2}}\,,$$
$$z(t) = e^{t^3}\,(\,12\,e^{2t^3}-11\,)^{-\frac{3}{2}}\,,$$

Therefore, the solution of fuzzy Bernoulli differential equation (14) with [ii.gH]-differentiability can be determined as follow:

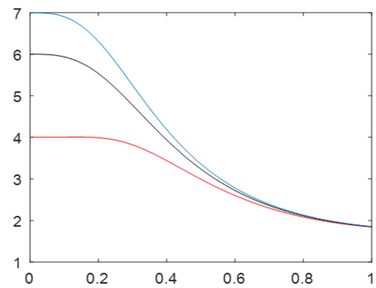
$$\tilde{u}(t) = \Big(6\,e^{t^3}\,(\,12e^{2t^3}-11\,)^{-\frac{1}{2}},\, 2\,e^{t^3}\,(\,12\,e^{2t^3}-11\,)^{-\frac{3}{2}},\, e^{t^3}\,(\,12\,e^{2t^3}-11\,)^{-\frac{3}{2}}\Big)\,.$$

The results are presented in Figure 4.

Example 4.3. Let a fuzzy Bernoulli differential equation be as follow:

$$u'(t) + (-t-1)\,\tilde{u}(t) = (2\,t\,+2)\,\tilde{u}^2(t) \qquad \tilde{u}(0) = (-4, 3, 3)\,, \tag{14}$$

Figure 4: [ii.gH]-differentiable solution of equation (13).



Solving 1-cut of the system in the first step results to the following equation:

$$x\,(t) = 4\,e^{\frac{t^2}{2}+t}\,\left(\,-\,8\,e^{\frac{t^2}{2}+t}+7\,\right)^{-1},$$

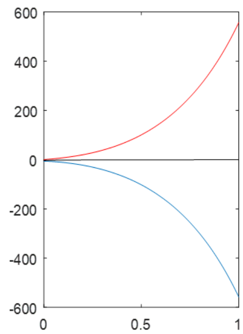
It is obvious that $x(t)$ is negative and continues and $p\,(t) < 0, q\,(t) > 0$. Hence, in the second step, [i.gH]-differentiability of theorem 3.3 part D1 is used to have the following results:

$$y\,(t) = \,e^{\frac{t^2}{2}+t}\,\left(\,-\,8\,e^{\frac{t^2}{2}+t}+7\,\right)^{-2}\left(\,3\,e^{-t^2-2t}\left(\,-\,10976\,e^{\frac{t^2}{2}+t}-14336\,e^{\frac{3}{2}t^2+3t}\right.\right.$$
$$\left.\left.+4096\,e^{2t^2+4t}+2401\,\right)+56448\,\right)\,,$$
$$z\,(t) = \,e^{\frac{t^2}{2}+t}\,\left(\,-\,8\,e^{\frac{t^2}{2}+t}+7\,\right)^{-2}\left(\,3\,e^{-t^2-2t}\,\left(\,-\,10976\,\,e^{\frac{t^2}{2}+t}-14336\,e^{\frac{3}{2}t^2+3t}\right.\right.$$
$$\left.\left.+4096\,e^{2t^2+4t}+2401\,\right)+56448\,\right)\,,$$

It is found that $y\,(t) > 0, z\,(t) > 0$. As a result, the solution of fuzzy Bernoulli differential equation (14) using [i.gH]-differentiability is achieved as follow and is drawn in Figure 5:

$$\tilde{u}\,(t) = \left(4\,e^{\frac{t^2}{2}+t}\,\left(\,-\,8e^{\frac{t^2}{2}+t}+7\,\right)^{-1}\,,e^{\frac{t^2}{2}+t}\,\left(\,-\,8\,e^{\frac{t^2}{2}+t}+7\,\right)^{-2}\left(\,3\,e^{-t^2-2t}\right.\right.$$
$$\left(\,-\,10976\,e^{\frac{t^2}{2}+t}-14336\,e^{\frac{3}{2}t^2+3t}+4096\,e^{2t^2+4t}+2401\,\right)+56448\,\right)\,,e^{\frac{t^2}{2}+t}$$
$$\left.\left(-8\,e^{\frac{t^2}{2}+t}+7\,\right)^{-2}\left(\,3\,e^{-t^2-2t}\left(\,-\,10976\,e^{\frac{t^2}{2}+t}-14336\,e^{\frac{3}{2}t^2+3t}+4096\,e^{2t^2+4t}+2401\,\right)+56448\right)\,,$$

Figure 5: [i.gH]-differentiable solution of equation (14).



As was expected, since the initial condition of equation (14) is symmetric, FBDE has fuzzy symmetric solution. Similarly, using [ii.gH]-differentiability of theorem 3.3 part A3, the following results can be obtained:

$$y\,(t) = 3\,e^{\frac{t^2}{2}+t}\;(\,-\,8e^{\frac{t^2}{2}+t}+7\,)^{-1}\,,$$

$$z\,(t) = 3\,e^{\frac{t^2}{2}+t}\;(\,-\,8e^{\frac{t^2}{2}+t}+7\,)^{-1}\,.$$

Moreover, it can be found that fuzzy Bernouli differential equation (14) does not have [ii.gH]-differentiable solution.

5 Conclusion

Fuzzy Bernoulli differential equations can be considered as one of the most important math equations. To solve such equations, an iterative method is proposed by Behzadi under generalized H-differentiability condition in case where the fuzzy variables are parametric as in [19]. In the present paper, an analytical method is applied to solve fuzzy Bernoulli differential equation in case where the fuzzy variables are LR fuzzy function. It is assumed that the generalized Hakuhara difference and the derivative of the fuzzy function exist. The significant advantage of the proposed method is its simplicity and applicability which enables it to be extended to all fuzzy functions. In the future work, fully fuzzy Bernoulli differential equation are going to be solved and the problem of switching point are going to be considered.

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References

[1] .S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (1987), 319-330.

[2] S. James, J. Buckley and T. Feuring, Fuzzy differential equations,F uzzy Sets and Systems 110 (2000), 43-54.

[3] D. Vorobiev, S. Seikkala, Towards the theory of fuzzy differential equations, Fuzzy Sets and Systems 125 (2002), 231-237.

[4] T. Gnana Bhaskar, V. Lakshmikantham and V. Devi, Revisiting fuzzy differential equations, Nonlinear Analysis 58 (2004), 351- 358.

[5] L. Zadeh, Toward a generalized theory of uncertainty (GTU) an outline, Information Sciences 172 (2005), 1-40.

[6] S.Ch. Palligkinis, G. Papageorgiou and I.Th. Famelis, RungeKutta, methods for fuzzy differential equations, Applied Mathematics and Computation209 (2009), 97-105.

[7] T. Allahviranloo, Z. Gouyandeh and A. Armand, A full fuzzy method for solving differential equation based on Taylor expansion, Journal of Intelligent & Fuzzy System 29 (2015), 1039-1055.

[8] M. Pakdaman, A. Ahmadian, S. Effati, S. Salahshour and D. Baleanu, Solving differential equations of fractional order using an optimization technique based on training artificial neural network, Applied Mathematics and Computation 293 (2017), 81-95.

[9] M. Puri, D. Ralescu, Differentials of fuzzy functions, Journal of Mathematical Analysis and Applications 91 (1983) 552-558.

[10] B.Bede,S.G. Gal, Generalizations of the differentiability of fuzzy number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems 151 (2005), 581-599.

[11] B. Bede, T. GnanaBhaskar, V. Lakshmikantham, Perspectives of Fuzzy Initial Value Problems, Communications in Applied Analysis 11 (2007), 339-357.

[12] O. A. Arqub, Series solution of fuzzy differential equations under strongly generalized differentiability, journal of Advanced Research in Applied Mathematics 5 (2013), 31-52.

[13] M. Mosleh, M. Otadi, Approximate solution of fuzzy differential equations under generalized differentiability, Applied Mathematical Modeling 39 (2015), 10-11.

[14] V. Nirmala, Numerical Solution of Fuzzy Differential Equations under generalized differentiability concept by New Runge-Kutta –Like Formula of Order 4, International Journal of Applied Engineering 10 (2018), Research 10.

[15] H.J. Zimmermann, Fuzzy Set Theory and its Applications, second, Kluwer Academic, Norwell, 1996.

[16] D. Dubois, H. Prade, Systems of linear fuzzy constraints, Fuzzy Sets Syst. 3 (1980) 37-48.

[17] B. Bede and L. Stefanini, Generalized differentiability of fuzzy valued functions, Fuzzy Sets and Systems 230 (2013), 119–141.

[18] L.Stefanini and B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, Nonlinear Analysis 71 (2009), 1311–1328.

[19] Sh. Sadigh Behzadi, Application of iterative method for solving fuzzy Bernoulli equation under generalized H-differentiability, Int. J. Industrial Mathematics6 (2014), 8 pages.

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